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Note

A Note on Simultaneous Polynomial Approximation in $L_p[-1, 1]$, $0 < p < 1$

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In this note we will show that for $0 < p < 1$ simultaneous polynomial approximation is not possible.

While I believe that the statement of the abstract is evident from the pathological behaviour of derivatives in L_p, $0 < p < 1$, numerous questions by others led me to understand the need for the following precise statement and proof.

THEOREM 1. *For $0 < p < 1$ we cannot have $P_n = P_n(f) \in \Pi_n$ such that*

$$\|f - P_n\|_p \leqslant C\omega^2(f, 1/n)_p \tag{1}$$

and

$$\|f' - P'_n\|_p \leqslant C_1 \omega(f', 1/n)_p \tag{2}$$

simultaneously with constants C, C_1 independent of n and $f \in$ A.C. $[-1, 1]$.

We recall that $\omega(g, t)_p = \sup_{|h| \leqslant t}(\|g(\cdot + (h/2)) - g(\cdot - (h/2))\|_{L_p[-1+h/2, 1-h/2]})$, $\omega^2(g, t)_p = \sup_{|h| \leqslant t}(\|g(\cdot + h) - 2g(\cdot) + g(\cdot - h)\|_{L_p[-1+h, 1-h]})$, and that $\|g\|_p = \|g\|_{L_p[-1, 1]}$ where $\|g\|_{L_p[a, b]} = (\int_a^b |g|^p)^{1/p}$.

COROLLARY 2. *Theorem 1 is valid if we replace (1) and (2) by*

$$\|f - P_n\|_p \leqslant C\omega_\varphi^2(f, 1/n)_p \tag{1'}$$

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and

$$\|f' - P_n'\|_p \leqslant C\omega_\varphi(f', 1/n)_p \tag{2'}$$

where $\varphi^2(x) = 1 - x^2$ *and* $\omega_\varphi^r(f, t)$ *is defined in* [1].

The corollary follows from the theorem as

$$\omega_\varphi^2(f, t)_p \leqslant C\omega^2(f, t)_p \qquad \text{and} \qquad \omega_\varphi(f', t)_p \leqslant C\omega(f', t)_p$$

(see for instance [2]).

Proof. Given p, C, and C_1, we construct f (that depends on n), which will cause a contradiction. Let $f(x)$ be given by

$$f(x) = \begin{cases} \dfrac{k+1}{n^2}, & \dfrac{k}{n^2} + \dfrac{1}{n^3} \leqslant x \leqslant \dfrac{k+1}{n^2}, \\ \dfrac{k}{n^2} + n\left(x - \dfrac{k}{n^2}\right), & \dfrac{k}{n^2} \leqslant x \leqslant \dfrac{k}{n^2} + \dfrac{1}{n^3}, \end{cases}$$

for some n ($n \geqslant 2$). Obviously,

$$f'(x) = \begin{cases} 0, & \dfrac{k}{n^2} + \dfrac{1}{n^3} < x < \dfrac{k+1}{n^3} \\ n, & \dfrac{k}{n^2} < x < \dfrac{k}{n^2} + \dfrac{1}{n^3}. \end{cases}$$

We now have

$$\omega^2(f, 1/n)_p^p = \omega^2(f - x, 1/n)_p^p \leqslant 4\,\|f - x\|_p^p \leqslant 4 \cdot 2\,\frac{1}{n^{2p}} = 8\,\frac{1}{n^{2p}}$$

and

$$\omega(f', 1/n)_p^p \leqslant 2\,\|f'\|_p^p = 2\,\frac{2n^2}{n^3}\,n^p = 4n^{p-1}.$$

Using now

$$\|f - P_n\|_p \leqslant C8^{1/p}n^{-2} \qquad \text{and} \qquad \|f' - P_n'\|_p \leqslant C_1 4^{1/p} n^{1-1/p}$$

in conjuction with

$$\|f - x\|_p \leqslant 2^{1/p}n^{-2} \qquad \text{and} \qquad \|f' - 0\|_p \leqslant 2^{1/p}n^{1-1/p},$$

we have

$$\|P_n - x\|_p \leqslant C_2 n^{-2} \qquad \text{and} \qquad \|P_n' - 0\|_p \leqslant C_3 n^{1-1/p}.$$

Using the Markov–Bernstein theorem (see [3, Theorem 5]),

$$\|P_n - x\|_p \leqslant C_2 n^{-2}$$

implies

$$\|\varphi(P'_n - 1)\|_p \leqslant C_4 n^{-1} \qquad \text{where} \quad \varphi^2 = 1 - x^2.$$

Hence

$$\left\{\int_{-1/2}^{1/2} |P'_n(x) - 1|^p \, dx\right\}^{1/p} \leqslant \sqrt{\frac{4}{3}}\, C_4 n^{-1}$$

and

$$\left\{\int_{-1/2}^{1/2} |P'_n(x)|^p \, dx\right\}^{1/p} \leqslant C_3 n^{1-1/p}.$$

This implies

$$1 = \int_{-1/2}^{1/2} 1 \, dx \leqslant C(n^{-p} + n^{p-1})^{1/p} = o(1), \qquad n \to \infty,$$

which is a contradiction. ▮

We note that (1) (or (1′)) and

$$\|f' - Q_n\|_p \leqslant C_1 \omega(f', 1/n)_p$$

can be proved, but not with $Q_n = P'_n$.

References

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